Global weak solutions for a three-component Camassa-Holm system with N-peakon solutions

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Abstract

In this paper we mainly investigate the Cauchy problem of a three-component Camassa-Holm system. By using the method of approximation of smooth solutions, a regularization technique and the special structure of the system, we prove the existence of global weak solutions to the system.

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Contents

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1 Introduction

In this paper we consider the Cauchy problem for the following three-component Camassa-Holm equations with N-peakon solutions:

$$(1.1)\qquad \begin{cases} u_t = -va_x + u_x b + \frac{3}{2}ub_x - \frac{3}{2}u(a_x c_x - ac), \\ v_t = 2vb_x + v_x b, \\ w_t = -vc_x + w_x b + \frac{3}{2}wb_x + \frac{3}{2}w(a_x c_x - ac), \\ u = a - a_{xx}, \\ v = \frac{1}{2}(b_{xx} - 4b + a_{xx}c_x - c_{xx}a_x + 3a_x c - 3ac_x), \\ w = c - c_{xx}, \\ u|_{t=0} = u_0, \quad v|_{t=0} = v_0, \quad w|_{t=0} = w_0. \end{cases}$$

This system was proposed by Geng and Xue in [26]. It is based on the following spectral problem

$$(1.2)\qquad \phi_x = U\phi, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 & 0 \\ 1+\lambda v & 0 & u \\ \lambda w & 0 & 0 \end{pmatrix},$$

where $u,\ v,\ w$ are three potentials and λ is a constant spectral parameter. It was shown in [26] that the N-peakon solitons of the system (1.1) have the form

$$(1.3)\qquad \begin{aligned} a(t,x) &= \sum_{i=0}^{N} a_i(t) e^{-|x-x_i(t)|}, \\ b(t,x) &= \sum_{i=0}^{N} b_i(t) e^{-2|x-x_i(t)|}, \\ c(t,x) &= \sum_{i=0}^{N} c_i(t) e^{-|x-x_i(t)|}, \end{aligned}$$

where $a_i,\ b_i,\ c_i$ and x_i evolve according to a dynamical system. Moreover, the author derived infinitely many conservation laws of the system (1.1). By setting $a = c = 0$, the system (1.1) reduces to

$$(1.4)\qquad v_t = 2v_x b + vb_x, \quad v = \frac{1}{2}(b_{xx} - 4b).$$

Taking advantage of an appropriate scaling $\widetilde{v}(t,x) = v(\frac{t}{2}, \frac{x}{2}),\ \ \widetilde{b}(t,x) = -b(t, \frac{x}{2})$, one can deduce that

$$(1.5)\qquad \widetilde{v}_t + 2\widetilde{v}_x \widetilde{b} + \widetilde{v}\widetilde{b}_x = 0, \quad \widetilde{v} = \widetilde{b} - \widetilde{b}_{xx},$$

which is nothing but the famous Camassa-Holm (CH) equation [4, 15]. The Camassa-Holm equation was derived as a model for shallow water waves [4, 15]. It has been investigated extensively because of its great physical significance in the past two decades. The CH equation has a bi-Hamiltonian structure [6, 21] and is completely integrable [4, 7]. The solitary wave solutions of the CH equation were considered in [4, 5], where the authors showed that the CH equation possesses peakon solutions of the form $Ce^{-|x-Ct|}$. It is worth mentioning that the peakons are solitons and their shape is alike that of the travelling water waves of greatest height, arising as solutions to the free-boundary problem for incompressible Euler equations over a flat bed (these being the governing equations for water waves), cf. the discussions in [9, 13, 14, 37]. Constantin and Strauss verified that the peakon solutions of the CH equation are orbitally stable in [17].

The local well-posedness for the CH equation was studied in [10, 11, 19, 34]. Concretely, for initial profiles $\widetilde{b}_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, it was shown in [10, 11, 34] that the CH equation has a unique solution in $C([0,T); H^s(\mathbb{R}))$. Moveover, the local well-posedness for the CH equation in Besov spaces $C([0,T); B^s_{p,r}(\mathbb{R}))$ with $s > \max(\frac{3}{2}, 1+\frac{1}{p})$ was proved in [19]. The global existence of strong solutions were established in [8, 10, 11] under some sign conditions and it was shown in [8, 10, 11, 12] that the solutions will blow up in finite time when the slope of initial data was bounded by a negative quantity. The global weak solutions for the CH equation were studied in [16] and [38]. The global conservative and dissipative solutions of CH equation were presented in [2] and [3], respectively.

A natural idea is to extend such study to the multi-component generalized systems. One of the most popular generalized systems is the following integrable two-component Camassa-Holm shallow water system (2CH) [18]:

$$\begin{cases} m_t + um_x + 2u_x m + \sigma\rho\rho_x = 0, \\ \rho_t + (u\rho)_x = 0, \end{cases} \tag{1.6}$$

where $m = u - u_{xx}$ and $\sigma = \pm 1$. Local well-posedness for (2CH) with the initial data in Sobolev spaces and in Besov spaces was established in [18], [20], and [27], respectively. The blow-up phenomena and global existence of strong solutions to (2CH) in Sobolev spaces were obtained in [20], [22] and [27]. The existence of global weak solutions for (2CH) with $\sigma = 1$ was investigated in [24].

The other one is the modified two-component Camassa-Holm system (M2CH) [28]:

$$\begin{cases} m_t + um_x + 2u_x m + \sigma\rho\overline{\rho}_x = 0, \\ \rho_t + (u\rho)_x = 0, \end{cases} \tag{1.7}$$

where $m = u - u_{xx}$, $\rho = (1-\partial_x^2)(\overline{\rho} - \overline{\rho}_0)$ and $\sigma = \pm 1$. Local well-posedness for (M2CH) with the initial data in Sobolev spaces and in Besov spaces was established in [23] and [39] respectively. The blow

up phenomena of strong solutions to (M2CH) were presented in [23]. The existence of global weak solutions for (M2CH) with $\sigma = 1$ was investigated in [25]. The global conservative and dissipative solutions of (M2CH) were studied in [35] and [36], respectively.

Recently, the authors in [31] studied the local well-posedness and global existence of strong solutions to (1.1) under some sign condition. However, the solitons of (1.1) are not strong solutions and do not belong to the spaces $H^s(\mathbb{R})$, $s > \frac{3}{2}$. This fact motivates us to study weak solutions of (1.1). The main idea is based on the approximation of the initial data by smooth functions producing a sequence of global strong solutions (a^n, b^n, c^n) of (1.1). This method was first utilized by Constantin and Molinet in [16]. Due that the structure of the system (1.1) is more complex than that of the CH equation, we can not obtain the desired result under the same condition mentioned in [16]. In order to obtain the existence of global weak solutions of (1.1), we have to assume that the initial data $(u_0, w_0) \in (L^1(\mathbb{R}) \cap L^{1+\varepsilon}(\mathbb{R}))$, for some $\varepsilon > 0$. The main difficulty is to get the uniform boundedness of b^n. In order to overcome this difficulty, we make good use of the special structure of the system.

The paper is organized as follows. In Section 2, we recall some properties about strong solutions of (1.1). Moveover, we give some a prior estimates which are crucial to prove our main result. In Section 3, we introduce the definition of weak solutions to (1.1) and then prove the global existence of weak solutions to (1.1).

2 Preliminaries

In this section we recall the global existence of strong solutions to (1.1) and some lemmas that will be used to prove our main result.

Lemma 2.1. *[31] Assume that $v_0 = 0$, $(u_0, w_0) \in (H^3(\mathbb{R}))^2$, and that $u_0 = a_0 - a_{0,xx}$ and $w_0 = c_0 - c_{0,xx}$ are nonnegative. Then the initial value problem (1.1) has a unique solution $(u,\ 0,\ w) \in [C(\mathbb{R}_+; H^3(\mathbb{R})) \cap C^1(\mathbb{R}_+; H^2(\mathbb{R}))]^3$. Moreover, $H_1(t) = \int_{\mathbb{R}} ac + a_x c_x$ and $H_2(t) = \int_{\mathbb{R}} uc_x dx = -\int_{\mathbb{R}} wa_x$ are conservation laws. For every $t \geq 0$ we have*

$$(1)\quad |a_x(t,x)| \leq a(t,x) \quad \text{and} \quad |c_x(t,x)| \leq c(t,x), \quad \forall x \in \mathbb{R},$$

$$(2)\quad u(t,x) \geq 0 \quad \text{and} \quad w(t,x) \geq 0, \quad \forall x \in \mathbb{R},$$

$$(3)\quad \|a_x(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \|a(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq C\|a(t,\cdot)\|_{H^1(\mathbb{R})} \leq C\exp\left[(4H_1(0) + H_2(0))t\right] \quad \text{and}$$

$$\|c_x(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \|c(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq C\|c(t,\cdot)\|_{H^1(\mathbb{R})} \leq C\exp\left[(4H_1(0) + H_2(0))t\right],$$

$$(4)\quad \|b(t,\cdot)\|_{L^\infty(\mathbb{R})}, \|b_x(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq H_1(0) + \frac{1}{4}H_2(0) + \exp\left[(8H_1(0) + 2H_2(0))t\right].$$

Lemma 2.2. *Assume that $v_0 = 0$, $(u_0, w_0) \in (H^3(\mathbb{R}))^2$, and that $u_0 = a_0 - a_{0,xx}$ and $w_0 = c_0 - c_{0,xx}$ are nonnegative. And let $(u, 0, w)$ be the corresponding solution to (1.1) as in Lemma 2.1. Then for any $t \in [0, T]$, there exists a constant C such that*

$$\|b(t,\cdot)\|_{H^1} \leq C(H_1(0) + H_2(0)) + C\exp\left[(8H_1(0) + 2H_2(0))t\right].$$

Proof. Since $v = 0$, it follows from (1.1) that $4b - b_{xx} = a_{xx}c_x - c_{xx}a_x + 3a_xc - 3ac_x$. Note that $G_2 * f = (4 - \partial_{xx})^{-1} f$ with $G_2(x) = \frac{1}{8}e^{-2|x|}$. Applying Young's inequality, we deduce that

$$\begin{aligned}
\|b(t,\cdot)\|_{H^1} &\leq C\int_{\mathbb{R}} |a_{xx}c_x - c_{xx}a_x + 3a_xc - 3ac_x|dx \\
&\leq C\int_{\mathbb{R}} (|u(c_x + c)| + |w(a_x + a)| + |uc| + |wa| + 2|a_xc| + 2|ac_x|)dx \\
&\leq C\int_{\mathbb{R}} (|u(c_x + c)| + |w(a_x + a)| + |uc| + |wa|)dx + +C\|a\|_{H^1}\|c\|_{H^1}.
\end{aligned}$$

Thanks to Lemma 2.1, we see that $u \geq 0$, $w \geq 0$, $a_x + a \geq 0$, $c_x + c \geq 0$, $a \geq 0$, $c \geq 0$, which leads to

$$\begin{aligned}
\|b(t,\cdot)\|_{H^1} &\leq C\int_{\mathbb{R}} [u(c_x + c) + w(a_x + a) + uc + wa]dx + C\|a\|_{H^1}\|c\|_{H^1} \\
&\leq C(H_1(0) + H_2(0)) + C\exp\left[(8H_1(0) + 2H_2(0))t\right].
\end{aligned}$$

$\square$

Now we present some L^p-estimates of the strong solution to (1.1) where $p \in [1, \infty]$.

Lemma 2.3. *Assume that $v_0 = 0$, $(u_0, w_0) \in (H^3(\mathbb{R}))^2$, and that $u_0 = a_0 - a_{0,xx}$ and $w_0 = c_0 - c_{0,xx}$ are nonnegative and belong to $L^1(\mathbb{R}) \cap L^{1+\varepsilon}(\mathbb{R})$ for some $\varepsilon > 0$. And let $(u, 0, w)$ be the corresponding solution of (1.1) as in Lemma 2.1. Then for any $t \in [0, T]$, there exists a constant C_T such that*

(2.1)
$$\|a(t,\cdot)\|_{L^1(\mathbb{R})} = \|u(t,\cdot)\|_{L^1(\mathbb{R})} \leq e^{tC_T}\|u_0\|_{L^1(\mathbb{R})}, \quad \|c(t,\cdot)\|_{L^1(\mathbb{R})} = \|w(t,\cdot)\|_{L^1(\mathbb{R})} \leq e^{tC_T}\|w_0\|_{L^1(\mathbb{R})},$$

(2.2)
$$\|a(t,\cdot)\|_{L^{1+\varepsilon}(\mathbb{R})} \leq \|u(t,\cdot)\|_{L^{1+\varepsilon}(\mathbb{R})} \leq e^{tC_T}\|u_0\|_{L^{1+\varepsilon}(\mathbb{R})}, \quad \|c(t,\cdot)\|_{L^{1+\varepsilon}(\mathbb{R})} \leq \|w(t,\cdot)\|_{L^{1+\varepsilon}(\mathbb{R})} \leq e^{tC_T}\|u_0\|_{L^{1+\varepsilon}(\mathbb{R})}.$$

Proof. By density argument, we assume that $u(t,\cdot) \in C_0^\infty(\mathbb{R})$ and $w(t,\cdot) \in C_0^\infty(\mathbb{R})$. By virtue of (1.1) and integration by parts , we have

$$\begin{aligned}
(2.3)\quad \frac{d}{dt}\int_{-\infty}^{+\infty} u dx &= \int_{-\infty}^{+\infty} u_t dx = \int_{-\infty}^{+\infty} u_x b + \frac{3}{2}ub_x - \frac{3}{2}u(a_xc_x - ac)dx \\
&= \int_{-\infty}^{+\infty} \frac{1}{2}ub_x - \frac{3}{2}u(a_xc_x - ac)dx \leq \left\{\frac{1}{2}\|b_x\|_{L^\infty([0,T)\times\mathbb{R})} + \frac{3}{2}\|a_xc_x - ac\|_{L^\infty([0,T)\times\mathbb{R})}\right\}\|u\|_{L^1(\mathbb{R})}.
\end{aligned}$$

Taking advantage of Lemma 2.1 and using the fact that $u \geq 0$, we deduce that

$$\|u(t,\cdot)\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})} + \int_0^t C_T \|u(s,\cdot)\|_{L^1(\mathbb{R})} ds. \tag{2.4}$$

Applying Gronwall's inequality, we infer that

$$\|u(t,\cdot)\|_{L^1(\mathbb{R})} \leq e^{tC_T} \|u_0\|_{L^1(\mathbb{R})}. \tag{2.5}$$

Since $a(t,x)$ and $u(t,x)$ are nonnegative, it follows that

$$\|u(t,\cdot)\|_{L^1(\mathbb{R})} = \int_{-\infty}^{+\infty} u(t,x)dx = \int_{-\infty}^{+\infty} a(t,x) - a_{xx}(t,x)dx = \int_{-\infty}^{+\infty} a(t,x)dx = \|a(t,\cdot)\|_{L^1(\mathbb{R})}. \tag{2.6}$$

By the same token, we obtain

$$\|c(t,\cdot)\|_{L^1(\mathbb{R})} = \|w(t,\cdot)\|_{L^1(\mathbb{R})} \leq e^{tC_T} \|w_0\|_{L^1(\mathbb{R})}. \tag{2.7}$$

Now we turn our attention to prove (2.2). If $\varepsilon < \infty$, By virtue of (1.1) and integration by parts , we have

(2.8)
$$\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{+\infty} u^{1+\varepsilon} dx = (1+\varepsilon)\int_{-\infty}^{+\infty} u^{\varepsilon} u_t dx = \int_{-\infty}^{+\infty} u_x^{1+\varepsilon} b + \frac{3(1+\varepsilon)}{2} u^{1+\varepsilon} b_x - \frac{3(1+\varepsilon)}{2} u^{1+\varepsilon}(a_x c_x - ac) dx \\
&= \int_{-\infty}^{+\infty} \frac{1+3\varepsilon}{2} u^{1+\varepsilon} b_x - \frac{3(1+\varepsilon)}{2} u^{1+\varepsilon}(a_x c_x - ac) dx \\
&\leq \left\{ \frac{1+3\varepsilon}{2} \|b_x\|_{L^\infty([0,T)\times\mathbb{R})} + \frac{3(1+\varepsilon)}{2} \|a_x c_x - ac\|_{L^\infty([0,T)\times\mathbb{R})} \right\} \|u\|_{L^{1+\varepsilon}(\mathbb{R})}^{1+\varepsilon},
\end{aligned}$$

which along with $u \geq 0$ leads to

$$\begin{aligned}
\frac{d}{dt}\|u\|_{L^{1+\varepsilon}(\mathbb{R})} &\leq \left\{ \frac{1+3\varepsilon}{2(1+\varepsilon)} \|b_x\|_{L^\infty([0,T)\times\mathbb{R})} + \frac{3}{2} \|a_x c_x - ac\|_{L^\infty([0,T)\times\mathbb{R})} \right\} \|u\|_{L^{1+\varepsilon}(\mathbb{R})} \\
&\leq \frac{3}{2}(\|b_x\|_{L^\infty([0,T)\times\mathbb{R})} + \|a_x c_x - ac\|_{L^\infty([0,T)\times\mathbb{R})}) \|u\|_{L^{1+\varepsilon}(\mathbb{R})}.
\end{aligned} \tag{2.9}$$

Taking advantage of Lemma 2.1 and Gronwall's inequality, we infer that

$$\|u(t,\cdot)\|_{L^{1+\varepsilon}(\mathbb{R})} \leq e^{tC_T} \|u_0\|_{L^{1+\varepsilon}(\mathbb{R})}. \tag{2.10}$$

If $\varepsilon = \infty$, using a similar calculation for any $0 < \delta < \infty$, and then taking limit as $\delta \to \infty$, we obtain

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq e^{tC_T} \|u_0\|_{L^\infty(\mathbb{R})}. \tag{2.11}$$

Note that $G_1 * f = (1-\partial_{xx})^{-1} f$ with $G_1(x) = \frac{1}{2}e^{-|x|}$. Using Young's inequality, we deduce that

$$\|a(t,\cdot)\|_{L^{1+\varepsilon}(\mathbb{R})} = \|G_1 * u(t,\cdot)\|_{L^{1+\varepsilon}(\mathbb{R})} \leq \|u(t,\cdot)\|_{L^{1+\varepsilon}(\mathbb{R})} \leq e^{tC_T} \|u_0\|_{L^\infty(\mathbb{R})}. \tag{2.12}$$

By the same token, we get

$$\|c(t,\cdot)\|_{L^{1+\varepsilon}(\mathbb{R})} \le \|w(t,\cdot)\|_{L^{1+\varepsilon}(\mathbb{R})} \le e^{tC_T}\|w_0\|_{L^\infty(\mathbb{R})}. \tag{2.13}$$

□

Let us now recall a partial integration result for Bochner spaces.

Lemma 2.4. *[33] Let $T>0$. If*

$$f,\ g \in L^2(0,T;H^1(\mathbb{R})) \quad \textit{and} \quad \frac{df}{dt},\frac{dg}{dt} \in L^2(0,T;H^{-1}(\mathbb{R})),$$

then $f,\ g$ are a.e. equal to a function continuous from $[0,T]$ into $L^2(\mathbb{R})$ and

$$\langle f(t),g(t)\rangle - \langle f(s),g(s)\rangle = \int_s^t \langle \frac{df(\tau)}{d\tau}, g(\tau)\rangle d\tau + \int_s^t \langle \frac{dg(\tau)}{d\tau}, f(\tau)\rangle d\tau$$

for all $s,t\in[0,T]$, where $\langle\cdot,\cdot\rangle$ is the $H^{-1}(\mathbb{R})$ and $H^1(\mathbb{R})$ duality bracket.

Throughout this paper, let $\{\rho_n\}_{n\ge1}$ denote the mollifiers

$$\rho_n(x) = \left(\int_{\mathbb{R}} \rho(y)dy\right)^{-1} n\rho(nx), \quad x\in\mathbb{R}, \quad n\ge 1,$$

where $\rho\in C_0^\infty(\mathbb{R})$ is defined by

$$\rho(x) = \begin{cases} e^{\frac{1}{x^2-1}}, & |x|<1, \\ 0, & |x|>1. \end{cases}$$

3 Global weak solutions

In this section, we first introduce the definition of weak solutions to (1.1) with $v=0$. Note that $G_1 * f = (1-\partial_{xx})^{-1} f$ with $G_1(x) = \frac{1}{2}e^{-|x|}$. For smooth solutions of (1.1), we get

$$\begin{aligned} a_t &= G_1 * [u_x b + \frac{3}{2}u b_x - \frac{3}{2}u(a_x c_x - ac)] \\ &= a_x b + \frac{1}{2}\partial_x G_1 * (a_x b_x + a_x^2 c_x + 3ab - 3a_x ac) + \frac{1}{2}G_1 * (ba_x + 3a^2 c + 3aa_x c_x). \end{aligned} \tag{3.1}$$

By the same token, we obtain

$$\begin{aligned} c_t &= G_1 * [w_x b + \frac{3}{2}w b_x + \frac{3}{2}w(a_x c_x - ac)] \\ &= c_x b + \frac{1}{2}\partial_x G_1 * (b_x c_x - a_x c_x^2 + 3bc + 3acc_x) + \frac{1}{2}G_1 * (bc_x - 3ac^2 - 3a_x cc_x). \end{aligned} \tag{3.2}$$

For simplicity, we introduce the notation

$$f_1 = a_x b_x + a_x^2 c_x + 3ab - 3a_x ac, \quad f_2 = b_x c_x - a_x c_x^2 + 3bc + 3acc_x, \tag{3.3}$$

$$g_1 = ba_x + 3a^2c + 3aa_xc_x, \quad g_2 = bc_x - 3ac^2 - 3a_xcc_x. \tag{3.4}$$

Then (1.1) can be rewrite in the following hyperbolic type

$$\begin{cases} a_t = a_xb + \frac{1}{2}\partial_x G_1 * f_1 + \frac{1}{2}G_1 * g_1, \\ c_t = c_xb + \frac{1}{2}\partial_x G_1 * f_2 + \frac{1}{2}G_1 * g_2, \\ 4b - b_{xx} = a_{xx}c_x - c_{xx}a_x + 3a_xc - 3ac_x, \\ a|_{t=0} = a_0, \quad c|_{t=0} = c_0. \end{cases} \tag{3.5}$$

Definition 3.1. *Assume that $(a_0, c_0) \in (H^s(\mathbb{R}))^2$ with $s < \frac{5}{2}$. If $(a, c) \in [L^\infty_{loc}(0, T; H^s(\mathbb{R}))]^2$ and satisfies*

$$\int_0^T \int_{\mathbb{R}} (a\phi_t + a_xb\phi - \frac{1}{2}G_1 * f_1\phi_x + \frac{1}{2}G_1 * g_1\phi)dxdt + \int_{\mathbb{R}} a_0\phi(0, x)dx = 0, \quad \forall \phi \in C_0^\infty((-T, T) \times \mathbb{R}),$$

$$\int_0^T \int_{\mathbb{R}} (c\varphi_t + c_xb\varphi - \frac{1}{2}G_1 * f_2\varphi_x + \frac{1}{2}G_1 * g_2\varphi)dxdt + \int_{\mathbb{R}} c_0\varphi(0, x)dx = 0, \quad \forall \varphi \in C_0^\infty((-T, T) \times \mathbb{R}),$$

$$\int_{\mathbb{R}} (b(t,x)\psi - b(t,x)\psi_{xx})dx = \int_{\mathbb{R}} (a_{xx}c_x - c_{xx}a_x + 3a_xc - 3ac_x)(t,x)\psi dx = 0, \quad \text{for a.e. } t \in [0, T), \quad \forall \psi \in C_0^\infty(\mathbb{R}),$$

then (a, b, c) is called a weak solution to (3.5). Moreover, if $(a(t, x), b(t, x), c(t, x))$ is a weak solution on $[0, T)$ for any $T > 0$, then it is called a global weak solution to (3.5).

Our main result can be stated as follow.

Theorem 3.2. *Let $(a_0, c_0) \in H^1(\mathbb{R})$. Moreover $u_0 = a_0 - a_{0,xx}$ and $w_0 = c_0 - c_{0,xx}$ belong to $L^1(\mathbb{R}) \cap L^{1+\varepsilon}(\mathbb{R})$ for some $\varepsilon > 0$. If $u_0 \geq 0$ and $w_0 \geq 0$ a.e. on $\mathbb{R}$, then (3.5) has a global weak solution $(a, c) \in [W^{1,\infty}([0, T) \times \mathbb{R}) \cap C([0, T); L^2(\mathbb{R})) \cap C_w(0, T; H^1(\mathbb{R}))]^2$ for arbitrary finite $T > 0$. Moreover, $(u, w) \in [L^\infty_{loc}(\mathbb{R}_+; L^1(\mathbb{R}) \cap L^{1+\varepsilon}(\mathbb{R}))]^2$.*

Proof. **Step 1.** Without loss of generality, we assume that $\varepsilon < \infty$. Define $a_0^n = \rho_n * a_0 \in H^\infty(\mathbb{R})$ and $c_0^n = \rho_n * c_0 \in H^\infty(\mathbb{R})$ for $n \geq 1$. Then we have

$$a_0^n \to a_0 \quad \text{and} \quad c_0^n \to c_0 \quad \text{in} \quad H^1(\mathbb{R}), \quad \text{as} \quad n \to \infty. \tag{3.6}$$

Since $u_0^n = a_0^n - a_{0,xx}^n = \rho_n * u_0$ and $w_0^n = c_0^n - c_{0,xx}^n = \rho_n * w_0$ for $n \geq 1$, it follows that

$$u_0^n \to u_0 \quad \text{and} \quad w_0^n \to w_0 \quad \text{in} \quad L^1(\mathbb{R}) \cap L^{1+\varepsilon}(\mathbb{R}), \quad \text{as} \quad n \to \infty. \tag{3.7}$$

Note that $u_0^n \geq 0$ and $w_0^n \geq 0$. By Lemma 2.1, we obtain that there exists a global strong solution $(u^n, 0, w^n)$ of (1.1) with the initial data $(u_0^n, 0, w_0^n)$. Moreover $(u^n, w^n) \in [C([0, \infty); H^s(\mathbb{R})) \cap$

$C^1([0,\infty);H^{s-1}(\mathbb{R})]^2$ for any $s\geq 3$ and $u^n=a^n-a^n_{xx}\geq 0$, $w^n=c^n-c^n_{xx}\geq 0$.

Step 2. For fixed $T>0$, by virtue of Lemmas 2.1-2.2, we have

$$\|a^n_x\|_{L^\infty([0,T]\times\mathbb{R})}\leq\|a^n\|_{L^\infty([0,T]\times\mathbb{R})}\leq C\|a^n\|_{L^\infty(0,T;H^1(\mathbb{R}))}\leq C\exp\left[(4H^n_1(0)+H^n_2(0))T\right], \tag{3.8}$$

$$\|c^n_x\|_{L^\infty([0,T]\times\mathbb{R})}\leq\|c^n\|_{L^\infty([0,T]\times\mathbb{R})}\leq C\|c^n\|_{L^\infty([0,T];H^1(\mathbb{R}))}\leq C\exp\left[(4H^n_1(0)+H^n_2(0))T\right], \tag{3.9}$$

$$\begin{aligned}&\|b^n\|_{L^\infty([0,T]\times\mathbb{R})},\ \|b^n_x\|_{L^\infty([0,T]\times\mathbb{R})}\leq H^n_1(0)+\frac{1}{4}H^n_2(0)+\exp\left[(8H^n_1(0)+2H^n_2(0))T\right],\\&\|b^n\|_{L^\infty([0,T];H^1(\mathbb{R}))}\leq C\{(H^n_1(0)+H^n_2(0))+\exp\left[(4H^n_1(0)+H^n_2(0))T\right]\},\end{aligned} \tag{3.10}$$

where $H^n_1(0)=\int_{\mathbb{R}}a^n_0c^n_0+a^n_{0,x}c^n_{0,x}$ and $H^n_2(0)=\int_{\mathbb{R}}u^n_0c^n_{0,x}$. Applying Cauchy-Schwarz's inequality and Young's inequality, we obtain

$$H^n_1(0)\leq\|a^n_0\|_{H^1(\mathbb{R})}+\|c^n_0\|_{H^1(\mathbb{R})}\leq\|a_0\|_{H^1(\mathbb{R})}+\|c_0\|_{H^1(\mathbb{R})}. \tag{3.11}$$

Since $w^n_0\geq 0$, it follows that

$$H^n_2(0)\leq\|u^n_0\|_{L^1(\mathbb{R})}\|c^n_{0,x}\|_{L^\infty(\mathbb{R})}\leq\|u^n_0\|_{L^1(\mathbb{R})}\|c^n_0\|_{L^\infty(\mathbb{R})}\leq\|u_0\|_{L^1(\mathbb{R})}\|c_0\|_{L^\infty(\mathbb{R})}. \tag{3.12}$$

Plugging (3.11)-(3.12) into (3.8)-(3.10), we verify that (a^n,c^n) is uniformly bounded in $[L^\infty(0,T;W^{1,\infty}(\mathbb{R}))\cap L^\infty(0,T;H^1(\mathbb{R}))]^2$ and b^n is uniformly bounded in $L^\infty(0,T;W^{1,\infty}(\mathbb{R}))\cap L^\infty(0,T;H^1(\mathbb{R}))$. By virtue of (3.5), we obtain

$$a^n_t=a^n_xb^n+\frac{1}{2}\partial_xG_1*f^n_1+\frac{1}{2}G_1*g^n_1, \tag{3.13}$$

where $f^n_1=a^n_xb^n_x+(a^n_x)^2c^n_x+3a^nb^n-3a^n_xa^nc^n$ and $g^n_1=b^na^n_x+3(a^n)^2c^n+3a^na^n_xc^n_x$.
Since (a^n,c^n) is uniformly bounded in $[L^\infty(0,T;W^{1,\infty}(\mathbb{R}))\cap L^\infty(0,T;H^1(\mathbb{R}))]^2$, it follows that a^n_t is uniformly bounded in $L^\infty((0,T)\times\mathbb{R})\cap L^2((0,T)\times\mathbb{R})$. Similarly, we deduce that c^n_t is uniformly bounded in $L^\infty((0,T)\times\mathbb{R})\cap L^2((0,T)\times\mathbb{R})$. Therefore, it has a subsequence such that

$$(a^{n_k},c^{n_k})\rightharpoonup(a,c),\quad *\text{ weakly in}\quad[W^{1,\infty}((0,T)\times\mathbb{R})\cap H^1((0,T)\times\mathbb{R})]^2\quad\text{as}\quad n_k\to\infty, \tag{3.14}$$

and

$$(a^{n_k},c^{n_k})\xrightarrow{n_k\to\infty}(a,c),\quad\text{a.e. on}\quad(0,T)\times\mathbb{R}, \tag{3.15}$$

for some $(a,c)\in[W^{1,\infty}((0,T)\times\mathbb{R})\cap H^1((0,T)\times\mathbb{R})]^2$. By virtue of Lemma 2.3, we see that

$$\|a^{n_k}_{xx}\|_{L^\infty(0,T;L^1(\mathbb{R})}\leq\|a^{n_k}\|_{L^\infty(0,T;L^1(\mathbb{R})}+\|u^{n_k}\|_{L^\infty(0,T;L^1(\mathbb{R})}\leq 2e^{TC_T}\|u_0\|_{L^1(\mathbb{R})}. \tag{3.16}$$

Differentiating (3.13) with respect to x yields that

$$a^n_{xt}=a^n_{xx}b^n+a^n_xb^n_x+\frac{1}{2}G_1*f^n_1-\frac{1}{2}f^n_1+\frac{1}{2}\partial_xG_1*g^n_1,$$

which along with Young's inequality leads to $\|a_{xt}^{n_k}\|_{L^\infty(0,T;L^1(\mathbb{R})} \le C_T$. Since $T<\infty$, it follows that

$$\mathbb{V}[a_x^{n_k}] = \|a_{xx}^{n_k}\|_{L^1((0,T)\times\mathbb{R})} + \|a_{xt}^{n_k}\|_{L^1((0,T)\times\mathbb{R})} \le C_T, \tag{3.17}$$

where $\mathbb{V}(f)$ is the total variation of $f\in BV([0,T]\times\mathbb{R})$. By Helly's theorem (See [32]), there exists a subsequence, denoted again by $a_x^{n_k}$, such that

$$a_x^{n_k} \xrightarrow{n_k\to\infty} \alpha, \quad \text{a.e. on} \quad (0,T)\times\mathbb{R}, \tag{3.18}$$

where $\alpha\in BV((0,T)\times\mathbb{R})$ with $\mathbb{V}(\alpha)\le C_T$. From (3.15) we have $a_x^{n_k}\xrightarrow{n_k\to\infty} a_x$ in $\mathcal{D}'((0,T)\times\mathbb{R})$. This enables us to identify α with a_x for a.e. $t\in(0,T)\times\mathbb{R}$. Therefore

$$a_x^{n_k} \xrightarrow{n_k\to\infty} a_x, \quad \text{a.e. on} \quad (0,T)\times\mathbb{R}, \tag{3.19}$$

and $\mathbb{V}(a_x)\le C_T$. By the same token, we deduce that

$$c_x^{n_k} \xrightarrow{n_k\to\infty} c_x, \quad \text{a.e. on} \quad (0,T)\times\mathbb{R}. \tag{3.20}$$

Note that $G_2 * f = (4-\partial_{xx})^{-1} f$ with $G_2(x)=\frac{1}{8}e^{-2|x|}$. By virtue of (3.5), we have

$$b^n = G_2 * (a_{xx}^n c_x^n - c_{xx}^n a_x^n + 3a_x^n c^n - 3a^n c_x^n) = G_2 * (a_x^n w^n - c_x^n u^n + 2a_x^n c^n - 2a^n c_x^n). \tag{3.21}$$

By (1.1), we deduce that

(3.22)
$$\begin{aligned} b_t^n &= G_2 * (a_{xt}^n w^n - c_{xt}^n u^n) + G_2 * (a_x^n w_t^n - c_x^n u_t^n) + 2G_2 * (a_{x,t}^n c^n - a^n c_{x,t}^n) + 2G_2 * (a_x^n c_t^n - a_t^n c_x^n) \\ &= I^n + II^n + III^n + IV^n. \end{aligned}$$

Since a_x^n, a_t^n, c_x^n and c_t^n are uniformly bounded in $L^\infty((0,T)\times\mathbb{R})\cap L^2((0,T)\times\mathbb{R})$, it follows from Young's inequality that [1]

$$\|IV^n\|_{L^\infty\cap L^2} \le 2(\|a_x^n\|_{L^\infty\cap L^2}\|c_t^n\|_{L^\infty\cap L^2} + \|c_x^n\|_{L^\infty\cap L^2}\|a_t^n\|_{L^\infty\cap L^2}) \le C_T. \tag{3.23}$$

We first consider the term I^n. By virtue of (3.5), we see that

(3.24)
$$\begin{aligned} I^n &= G_2 * [(a_{xx}^n b^n + a_x^n b_x^n + \frac{1}{2}G_1 * f_1^n - \frac{1}{2}f_1^n + \frac{1}{2}\partial_x G_1 * g_1^n)w^n \\ &\quad - (c_{xx}^n b^n + b_x^n c_x^n + \frac{1}{2}G_1 * f_2^n - \frac{1}{2}f_2^n + \frac{1}{2}\partial_x G_2 * g_2^n)u^n] \\ &= G_2 * [(a_{xx}^n b^n + \frac{1}{2}a_x^n b_x^n - \frac{1}{2}(a_x^n)^2 c_x^n - \frac{3}{2}a^n b^n + \frac{3}{2}a_x^n a^n c^n + \frac{1}{2}G_1 * f_1^n + \frac{1}{2}\partial_x G_1 * g_1^n)w^n] \end{aligned}$$

[1] For simplicity, we use the notation $\|\cdot\|_{L^\infty\cap L^2}$ instead of $\|\cdot\|_{L^\infty((0,T)\times\mathbb{R})\cap L^2((0,T)\times\mathbb{R})}$.

$$
\begin{aligned}
&- (c^n_{xx}b^n + \frac{1}{2}c^n_x b^n_x + \frac{1}{2}a^n_x (c^n_x)^2 - \frac{3}{2}b^n c^n - \frac{3}{2}a^n c^n c^n_x + \frac{1}{2}G_1 * f^n_2 + \frac{1}{2}\partial_x G_2 * g^n_2)u^n \\
&= G_2 * [(a^n_{xx}c^n - c^n_{xx}a^n)b^n] + \frac{1}{2}G_2 * [b^n_x(a^n_x w^n - c^n_x u^n)] - \frac{1}{2}G_2 * [(a^n_x c^n_x - 3a^n c^n)(a^n_x w^n + c^n_x u^n)] \\
&+ \frac{1}{2}G_2 * [(G_1 * f^n_1 + \partial_x G_1 * g^n_1)w^n] + \frac{1}{2}G_2 * [(G_1 * f^n_2 + \partial_x G_1 * g^n_2)u^n] \\
&= I^n_1 + I^n_2 + I^n_3 + I^n_4 + I^n_5.
\end{aligned}
$$

Bounds for I^n_1. Since $(a^n_{xx}c^n - c^n_{xx}a^n) = (a^n_x c^n - c^n_x a^n)_x$, it follows that

$$
I^n_1 = 4G_2 * [(a^n_x c^n - c^n_x a^n)_x b^n] = 4\partial_x G_2 * [(a^n_x c^n - c^n_x a^n)b^n] - 4G_2 * [(a^n_x c^n - c^n_x a^n)b^n_x], \tag{3.25}
$$

which together with Young's inequality leads to $\|I^n_1\|_{L^\infty \cap L^2} \le C_T$.

Bounds for I^n_2. In view of the fact that $a^n_x w^n - c^n_x u^n = 2a^n c^n_x - 2a^n_x c^n + 4b^n - b^n_{xx}$, which implies that

$$
\begin{aligned}
I^n_2 &= \frac{1}{2}G_2 * [(2a^n c^n_x - 2a^n_x c^n + 4b^n - b^n_{xx})b^n_x] \\
&= G_2 * [(a^n c^n_x - a^n_x c^n)b^n_x] + \partial_x G_2 * (b^n)^2 - \frac{1}{4}\partial_x G_2 * (b^n_x)^2,
\end{aligned} \tag{3.26}
$$

which along with Young's inequality leads to $\|I^n_2\|_{L^\infty \cap L^2} \le C_T$.

Bounds for I^n_3. Thanks to $(a^n_x w^n + c^n_x u^n) = (a^n c^n - a^n_x c^n_x)_x$, we deduce that

$$
\begin{aligned}
I^n_3 &= \frac{1}{2}\partial_x G_2 * [(a^n_x c^n_x - a^n c^n)^2] + 2G_2 * [a^n c^n (a^n_x c^n_x - a^n c^n)_x] \\
&= \frac{1}{2}\partial_x G_2 * [(a^n_x c^n_x - a^n c^n)^2] - 2G_2 * [(a^n c^n)_x (a^n_x c^n_x - a^n c^n)] + 2\partial_x G_2 * [a^n c^n (a^n_x c^n_x - a^n c^n)],
\end{aligned} \tag{3.27}
$$

which along with Young's inequality implies that $\|I^n_3\|_{L^\infty \cap L^2} \le C_T$.

Bounds for I^n_4 **and** I^n_5. Since $\partial_{xx} G_1 * f = G_1 * f - f$, it follows that

$$
\begin{aligned}
I^n_4 &= \frac{1}{2}G_2 * [(G_1 * f^n_1 + \partial_x G_1 * g^n_1)c^n] - \frac{1}{2}G_2 * [(G_1 * f^n_1 + \partial_x G_1 * g^n_1)c^n_{xx}] \\
&= \frac{1}{2}G_2 * [(G_1 * f^n_1 + \partial_x G_1 * g^n_1)c^n] - \frac{1}{2}\partial_x G_2 * [(G_1 * f^n_1 + \partial_x G_1 * g^n_1)c^n_x] \\
&+ \frac{1}{2}G_2 * [(\partial_x G_1 * f^n_1 + G_1 * g^n_1 - g^n_1)c^n_x].
\end{aligned} \tag{3.28}
$$

Note that f^n_1 and g^n_1 are bounded in $L^\infty((0,T)\times\mathbb{R}) \cap L^2((0,T)\times\mathbb{R})$. Taking advantage of Young's inequality yields that $\|I^n_4\|_{L^\infty \cap L^2} \le C_T$. By the same token, we have $\|I^n_5\|_{L^\infty \cap L^2} \le C_T$. Thus, we show that $\|I^n\|_{L^\infty \cap L^2} \le C_T$. Since the estimate for III^n is similar to that of I^n, we omit the details here. Now we turn our attention to estimate the term II^n. By virtue of (3.5), we have

$$
II^n = G_2 * \{a^n_x[w^n_x b^n + \frac{3}{2}w^n b^n_x + \frac{3}{2}w^n(a^n_x c^n_x - a^n c^n)] - c^n_x[u^n_x b^n + \frac{3}{2}u^n b^n_x - \frac{3}{2}u^n(a^n_x c^n_x - a^n c^n)]\} \tag{3.29}
$$

$$
\begin{aligned}
&= G_2 * [(a_x^n w_x^n - c_x^n u_x^n) b^n] + \frac{3}{2} G_2 * [(a_x^n w^n - c_x^n u^n) b_x^n] + \frac{3}{2} G_2 * [(w^n a_x^n + c_x^n u^n)(a_x^n c_x^n - a^n c^n)] \\
&= II_1^n + II_2^n + II_3^n.
\end{aligned}
$$

Since $a_x^n w_x^n - c_x^n u_x^n = a_{xxx}^n c_x^n - a_x^n c_{xxx}^n = (a_{xx}^n c_x^n - a_x^n c_{xx}^n)_x = (3a^n c_x^n - 3a_x^n c^n + 4b^n - b_{xx}^n)_x$, it follows that

(3.30)
$$
\begin{aligned}
II_1^n &= \frac{1}{2} G_2 * [(3a^n c_x^n - 3a_x^n c^n + 4b^n - b_{xx}^n)_x b^n] \\
&= \frac{3}{2} \partial_x G_2 * [(a^n c_x^n - a_x^n c^n) b^n] - \frac{3}{2} G_2 * [(a^n c_x^n - a_x^n c^n) b_x^n] + \partial_x G_2 * (b^n)^2 - \frac{1}{2} G_2 * (b_{xxx}^n b^n).
\end{aligned}
$$

From the above identity, it is sufficient to bound for $G_2 * (b_{xxx}^n b^n)$. Indeed,

$$
\begin{aligned}
G_2 * (b_{xxx}^n b^n) &= \partial_x G_2 * (b_{xx}^n b^n) - G_2 * (b_{xx}^n b_x^n) \\
&= \partial_{xx} G_2 * (b_x^n b^n) - \partial_x G_2 * (b_x^n)^2 - \frac{1}{2} \partial_x G_2 * (b_x^n)^2 \\
&= 4 G_2 * (b_x^n b^n) - b_x^n b^n - \frac{3}{2} \partial_x G_2 * (b_x^n)^2.
\end{aligned}
\tag{3.31}
$$

By virtue of Young's inequality, we obtain $\|II_1^n\|_{L^\infty \cap L^2} \le C_T$. By the similar estimates as for I_2^n and I_3^n, we infer that $\|II_2^n\|_{L^\infty \cap L^2}$, $\|II_3^n\|_{L^\infty \cap L^2} \le C_T$. From the above argument, we prove that b_t^n is bounded in $L^\infty((0,T) \times \mathbb{R}) \cap L^2((0,T) \times \mathbb{R})$. Moreover, there exists a subsequence such that

$$
b^{n_k} \rightharpoonup b, \quad * \text{ weakly in} \quad W^{1,\infty}((0,T) \times \mathbb{R}) \cap H^1((0,T) \times \mathbb{R}) \quad \text{as} \quad n_k \to \infty,
\tag{3.32}
$$

and

$$
b^{n_k} \xrightarrow{n_k \to \infty} b, \quad \text{a.e. on} \quad (0,T) \times \mathbb{R},
\tag{3.33}
$$

for some $b \in W^{1,\infty}((0,T) \times \mathbb{R}) \cap H^1((0,T) \times \mathbb{R})$.

By virtue of Young's inequality, we have

(3.34)
$$
\|b_{xx}^{n_k}\|_{L^1((0,T)\times\mathbb{R})} \le \|b^n(t,\cdot)\|_{L^1((0,T)\times\mathbb{R})} + \|(a_{xx}^n c_x^n - c_{xx}^n a_x^n + 3a_x^n c^n - 3a^n c_x^n)\|_{L^1((0,T)\times\mathbb{R})} \le C_T.
$$

By differentiating both sides of (3.22) with respect to x, we obtain that

$$
b_{t,x}^{n_k} = I_x^{n_k} + II_x^{n_k} + III_x^{n_k} + IV_x^{n_k}.
\tag{3.35}
$$

Thanks to $\partial_x G_2 \in L^p$ for any $1 \le p \le \infty$, one can follows the similar proof as bound for b_t^n to deduce that

$$
\|b_{tx}^{n_k}\|_{L^1((0,T)\times\mathbb{R})} \le C_T.
\tag{3.36}
$$

By the same token as $a_x^{n_k}$, we deduce that there exists a subsequence denoted again by b^{n_k}, such that

$$b_x^{n_k} \xrightarrow{n_k\to\infty} b_x, \quad \text{a.e. on} \quad (0,T)\times\mathbb{R}, \tag{3.37}$$

and $\mathbb{V}(b_x)\le C_T$. For any fixed $t\in(0,T)$, we have $f_1^n,\ f_2^n,\ g_1^n,\ g_2^n$ are uniformly bounded in $L^\infty(\mathbb{R})$. Therefore, there exists a subsequence such that

(3.38)
$$(f_1^{n_k}(t,\cdot),f_2^{n_k}(t,\cdot),g_1^{n_k}(t,\cdot),g_2^{n_k}(t,\cdot)) \rightharpoonup (\widetilde{f_1}(t,\cdot),\widetilde{f_2}(t,\cdot),\widetilde{g_1}(t,\cdot),\widetilde{g_2}(t,\cdot)), \quad * \text{ weakly in } \ [L^\infty(\mathbb{R})]^4 \ \text{ as } \ n_k\to\infty.$$

By virtue of (3.19), (3.20) and (3.37), we deduce that $(\widetilde{f_1},\widetilde{f_2},\widetilde{g_1},\widetilde{g_2})=(f_1,f_2,g_1,g_2)$ for a.e. $t\in(0,T)$. Since $G_1(x)\in L^1(\mathbb{R})$, it follows that

$$(G_1*f_1^{n_k},G_1*f_2^{n_k},G_1*g_1^{n_k},G_1*g_2^{n_k}) \xrightarrow{n_k\to\infty} (G_1*f_1,G_1*f_2,G_1*g_1,G_1*g_2). \tag{3.39}$$

Noticing that $(a^n,b^n,c^n)\in[C^1((0,T);H^\infty)]^3$ is the strong solution of (3.5), we have

(3.40)
$$\int_0^T\int_{\mathbb{R}}(a^{n_k}\phi_t+a_x^{n_k}b^{n_k}\phi-\frac{1}{2}G_1*f_1^{n_k}\phi_x+\frac{1}{2}G_1*g_1^{n_k}\phi)dxdt+\int_{\mathbb{R}}a_0^{n_k}\phi(0,x)dx=0, \quad \forall\phi\in C_0^\infty((-T,T)\times\mathbb{R}),$$

(3.41)
$$\int_0^T\int_{\mathbb{R}}(c^{n_k}\varphi_t+c_x^{n_k}b^{n_k}\varphi-\frac{1}{2}G_1*f_2^{n_k}\varphi_x+\frac{1}{2}G_1*g_2^{n_k}\varphi)dxdt+\int_{\mathbb{R}}c_0^{n_k}\varphi(0,x)dx=0, \quad \forall\varphi\in C_0^\infty((-T,T)\times\mathbb{R}),$$

Taking limit as $n_k\to\infty$ in the above identities, we obtain

$$\int_0^T\int_{\mathbb{R}}(a\phi_t+a_xb\phi-\frac{1}{2}G_1*f_1\phi_x+\frac{1}{2}G_1*g_1\phi)dxdt+\int_{\mathbb{R}}a_0\phi(0,x)dx=0, \quad \forall\phi\in C_0^\infty((-T,T)\times\mathbb{R}),$$

$$\int_0^T\int_{\mathbb{R}}(c\varphi_t+c_xb\varphi-\frac{1}{2}G_1*f_2\varphi_x+\frac{1}{2}G_1*g_2\varphi)dxdt+\int_{\mathbb{R}}c_0\varphi(0,x)dx=0, \quad \forall\varphi\in C_0^\infty((-T,T)\times\mathbb{R}).$$

Step 3. According to Definition 3.1, it is sufficient to prove that (a,b,c) satisfies that

$$\int_{\mathbb{R}}(b(t,x)\psi-b(t,x)\psi_{xx})dx=\int_{\mathbb{R}}(a_{xx}c_x-c_{xx}a_x+3a_xc-3ac_x)(t,x)\psi dx=0, \ \text{ for a.e. } \ t\in[0,T), \ \ \forall\psi\in C_0^\infty(\mathbb{R}).$$

By virtue of (3.15), (3.19), (3.20) and (3.33), we deduce that

(3.42)
$$\int_{\mathbb{R}}(b^{n_k}(t,x)\psi-b^{n_k}(t,x)\psi_{xx})dx \xrightarrow{n_k\to\infty} \int_{\mathbb{R}}(b(t,x)\psi-b(t,x)\psi_{xx})dx \ \text{ for a.e. } \ t\in[0,T), \ \ \forall\psi\in C_0^\infty(\mathbb{R}),$$

(3.43)
$$\int_{\mathbb{R}}(3a_x^{n_k}c^{n_k}-3a^{n_k}c_x^{n_k})(t,x)\psi dx \xrightarrow{n_k\to\infty} \int_{\mathbb{R}}(3a_xc-3ac_x)(t,x)\psi dx \ \text{ for a.e. } \ t\in[0,T), \ \ \forall\psi\in C_0^\infty(\mathbb{R}).$$

For any fixed $t \in (0, T)$, taking advantage of Lemma 2.3, we have

(3.44)
$$\|u^{n_k}(t,\cdot)\|_{L^{1+\varepsilon}} \le C_T \|u_0^{n_k}\|_{L^{1+\varepsilon}} \le C_T \|u_0\|_{L^{1+\varepsilon}}, \quad \|w^{n_k}(t,\cdot)\|_{L^{1+\varepsilon}} \le C_T \|w_0^{n_k}\|_{L^{1+\varepsilon}} \le C_T \|w_0\|_{L^{1+\varepsilon}},$$

which along with Young's inequality lead to

$$\|a_{xx}^{n_k}(t,\cdot)\|_{L^{1+\varepsilon}} \le C_T, \quad \|c_{xx}^{n_k}(t,\cdot)\|_{L^{1+\varepsilon}} \le C_T. \tag{3.45}$$

Therefore there exists a subsequence, denoted again by $(a_{xx}^{n_k}(t,\cdot), c_{xx}^{n_k}(t,\cdot))$, such that

$$(a_{xx}^{n_k}(t,\cdot), c_{xx}^{n_k}(t,\cdot)) \rightharpoonup (a_{xx}(t,\cdot), c_{xx}(t,\cdot)) \quad \text{in} \quad L^{1+\varepsilon}(\mathbb{R}). \tag{3.46}$$

Since $W_{loc}^{2,1+\varepsilon}(\mathbb{R}) \hookrightarrow\hookrightarrow W_{loc}^{1,\infty}(\mathbb{R})$, it follows that

$$(a_x^{n_k}(t,\cdot), c_x^{n_k}(t,\cdot)) \xrightarrow{n_k \to \infty} (a_x(t,\cdot), c_x(t,\cdot)) \quad \text{in} \quad L_{loc}^{\infty}(\mathbb{R}). \tag{3.47}$$

For any $\psi \in C_0^\infty(\mathbb{R})$, we have

$$\int_{\mathbb{R}} (a_{xx}^{n_k} c_x^{n_k} - a_{xx} c_x)\psi dx = \int_{\mathbb{R}} (a_{xx}^{n_k} - a_{xx}) c_x \psi dx + \int_{\mathbb{R}} a_{xx}^{n_k} (c_x^{n_k} - c_x)\psi dx. \tag{3.48}$$

Using the fact that $\|c_x(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \liminf_{n_k \to \infty} \|c_x^{n_k}(t,\cdot)\|_{L^\infty(\mathbb{R})} \le C_T$ and by virtue of (3.48), we deduce that

$$\lim_{n_k \to \infty} \int_{\mathbb{R}} (a_{xx}^{n_k} - a_{xx}) c_x \psi dx = 0. \tag{3.49}$$

Suppose that *Supp* $\psi \subseteq (-K, K)$ with $K \ge 0$. Then, we see that

$$\begin{aligned} \int_{\mathbb{R}} a_{xx}^{n_k} (c_x^{n_k} - c_x)\psi dx &= \int_{-K}^{K} a_{xx}^{n_k} (c_x^{n_k} - c_x)\psi dx \le \|a_{xx}^{n_k}(t,\cdot)\|_{L^1(\mathbb{R})} \|c_x^{n_k} - c_x\|_{L^\infty(-K,K)} \|\psi\|_{L^\infty} \\ &\le C_T \|c_x^{n_k} - c_x\|_{L^\infty(-K,K)} \to 0 \quad \text{as} \quad n_k \to \infty. \end{aligned} \tag{3.50}$$

Taking the limit as $n_k \to \infty$ in (3.50), we get $\lim_{n_k \to \infty} \int_{\mathbb{R}} (a_{xx}^{n_k} c_x^{n_k} - a_{xx} c_x)\psi dx = 0$. By the same token we have that $\lim_{n_k \to \infty} \int_{\mathbb{R}} (c_{xx}^{n_k} a_x^{n_k} - c_{xx} a_x)\psi dx = 0$. Since that T can be taken arbitrarily, we show that (a, b, c) is indeed a global weak solution of (3.5) and belongs to $[W^{1,\infty}((0,T) \times \mathbb{R})]^3$.

Step 4. Note that $(\partial_t a^{n_k}(t,\cdot), \partial_t b^{n_k}(t,\cdot), \partial_t c^{n_k}(t,\cdot))$ is uniformly bounded in $L^2(\mathbb{R})$ for any $t \in (0, T)$. Hence, the map $t \mapsto (a^{n_k}(t,\cdot), b^{n_k}(t,\cdot), c^{n_k}(t,\cdot)) \in (H^1(\mathbb{R})^3)$ is weakly equicontinuous on $[0, T]$. It follows from the Arzela-Ascoli theorem that $(a^{n_k}(t,\cdot), b^{n_k}(t,\cdot), c^{n_k}(t,\cdot))$ contains a subsequence, denoted again by $(a^{n_k}(t,\cdot), b^{n_k}(t,\cdot), c^{n_k}(t,\cdot))$ converges weakly in $[H^1(\mathbb{R})]^3$ uniformly in t. The limit function $(a, b, c) \in [C_w([0, T); H^1(\mathbb{R}))]^3$.

By virtue of Fatou's Lemma, we have

$$\|a_t(t,\cdot)\|_{L_T^\infty(L^2(\mathbb{R}))} \le \liminf_{n_k \to \infty} \|a^{n_k}(t,\cdot)\|_{L_T^\infty(L^2(\mathbb{R}))} \le C_T, \tag{3.51}$$

$$\|c_t(t,\cdot)\|_{L^\infty_T(L^2(\mathbb{R}))} \le \liminf_{n_k\to\infty} \|c^{n_k}(t,\cdot)\|_{L^\infty_T(L^2(\mathbb{R}))} \le C_T. \tag{3.52}$$

Taking advantage of Lemma 2.4, we see that $(a,c) \in C([0,T);L^2(\mathbb{R}))$. □

Remark 3.3. *By virtue of Lemma 2.1, we have used the conservation law* $H_2(t) = \int_{\mathbb{R}} uc_x dx = \int_{\mathbb{R}} u_0 c_0' dx$ *to obtain the desired estimates, which implies that* u_0 *at least belongs to* $L^1(\mathbb{R})$*. However, the additional condition* $(u_0,w_0) \in L^{1+\varepsilon}(\mathbb{R})$ *in Theorem 3.2 is technical and unnatural. How to get rid of this condition is still an open problem.*

Remark 3.4. *In view of an interpolation argument, one can obtain that the solution* (a,c) *of (3.5) belongs to* $C([0,T)\times\mathbb{R})$ *for arbitrary finite* $T>0$*.*

Remark 3.5. *The condition* $u_0 \ge 0$ *and* $w_0 \ge 0$ *in Theorem 3.2 can be replaced by* u_0 *and* w_0 *don't change sign. One can follow the similar step to get the global existence of weak solution to (3.5).*

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